THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. I, No. 11—November, 1902.

(Whole Series, Vol. XI, No. 11.)

CONTENTS:

On Collineations of Space which leave Invariant a Quadric Surface, ... *Helen B. Brewster.*

PUBLISHED BY THE UNIVERSITY,

Lawrence, Kan.

Price of this number, 30 cents.

Entered at the post-office in Lawrence as second-class matter.

KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. I, No. 11. NOVEMBER, 1902. WHOLE SERIES, VOL. XI, No. 11.

COLLINEATIONS OF SPACE WHICH LEAVE INVARIANT A QUADRIC SURFACE.

BY HELEN B. BREWSTER.

1. In an article published in the "Annals of Mathematics," second series, Vol. II, No. 4. Dr. Ruth G. Wood has discussed "Collineations of Space which Transform a Non-degenerate Quadric Surface into Itself." In his "Theorie der Transformationsgruppen," Vol. III, pp. 251–254, Lie has given a table of groups of transformations having the above property, these groups having been developed analytically. Prof. H. B. Newson has developed a theory of collineations of space, his work being synthetical.

The object of this paper is to follow the last-named method, and thereby determine a table of all groups of collineations of space which leave invariant a non-degenerate quadric surface. This method has been found to admit more thorough classification than that of Doctor Wood.

An acquaintance of the reader with Professor Newson's work has been assumed, the language used being practically in complete agreement with that of Newson's writings; but it has been found to be impracticable to attempt the same symbolic notation. No effort has been made to repeat proofs given by Newson, and references to his paper are frequent. My sincere thanks are due to Professor Newson for the careful consideration he has given this work and the time he has spent in the correction of errors.

A close reading of the following table of symbols will obviate the danger of vagueness in the language of the remaining work. The pre-exponent and subscript of a given group denote, respectively, the type and number of parameters of the group; the invariant figure is denoted by the symbols enclosed in the parentheses, while constants of the transformation are given as subscripts to the parentheses. Thus, ${}^{1}G_{3}(\mathrm{lmF})_{a}$ denotes a three-parameter group of type I, having for in-

variant figure a quadric surface and one generator of each system on that surface, together with a constant relation between the cross-ratios of the one-dimensional transformations along these generators. We have indicated the various parts of the invariant figure as follows:

T = transformation.
$^{ex}G_{sub}$ = group of type (ex.); parameters (sub.)
F = "Fläche," a non-degenerate quadric surface.
l, l'....; m, m'...; = generators of first and second systems of generators on the surface, respectively.
λ; μ = all generators of first and second systems on the surface, respectively.
n, n' = reciprocal polar lines with regard to the quadric.
t, t'.... = lines tangent to surface.
l, *m*.... = lines of invariant points in their respective positions as generators, etc.
A, B, C, D = points on the surface.
P = point without the surface.
p = tangent plane at *A* to surface.
s = polar plane to P with regard to surface.
K, K'.... = conics on surface.
$\bar{x}$ = variation of the portion indicated as x.
$\underline{\quad}$ = contains.

A.—Transformations of Six Types. Number and Groups.

§ 1. Type XI.

2. *Fundamental group.*—If we consider the invariant figure of a transformation of space of type XI (Newson's classification), we find that this consists of a straight line, together with all points on, and all planes through, that line; and a pencil of lines in each plane, the vertices of which are distinct, but all lying on the axis of the pencil of the invariant planes. These lines, then, form the generators of a system of quadric surfaces, on each of which are left invariant all the generators of one system, and the generator, common to all, of the other system; therefore, also, the common tangent plane. In this work we shall choose one of the quadric surfaces and neglect any changes that may occur on the other quadric surfaces of the system. This group we shall call $^{11}G_1(l\mu F)$—the only parameter being the constant of the parabolic, one-dimensional transformation along each of the generators of system μ. The one-dimensional transformation along *l* is identical.

3. *Number of transformations.*—Of these groups there are double infinity leaving invariant the chosen quadric surface, each group

leaving invariant, in addition, all the generators of one system and one generator of the other system; *i. e.*, there is one such group for each generator of the given quadric surface. Hence, there are $2\infty^2$ transformations of type XI, of space, leaving a quadric surface invariant.

4. *No higher groups.*—These $2\infty^2$ transformations do not form a two-parameter group; for if we take the resultant of two transformations, ${}^{11}T(l\mu F)$ and ${}^{11}T_1(l_1\mu F)$, in which the invariant figure of the one differs from that of the other only in the position of the generator l, we have along each generator of the system μ two one-dimensional parabolic transformations without common invariant point, which generally do not result in a parabolic transformation; hence, the resultant collineation of space cannot be of type XI. Again, if we consider the only other possible combination, ${}^{11}T(l\mu F)$ and ${}^{11}T_1(m\lambda F)$, one taken on each of the systems of generators, along m we have identical and parabolic one-dimensional transformations whose resultant is parabolic; along l we have again the resultant of identical and parabolic one-dimensional transformation, which, being parabolic, makes the resultant a collineation of type XIII.

THEOREM. There are $2\infty^2$ transformations of space of type XI, skew elations, which leave invariant a given quadric surface, each one of which leaves also invariant all of one system, and one of the second system of generators. These $2\infty^2$ transformations form $2\infty^1$ one-parameter groups, one for each generator of the surface, but they do not combine to form two-parameter groups.

§ 2. TYPE X.

5. *Fundamental group.*—In the fundamental group of type X, the invariant figure consists of two non-intersecting lines and the congruence meeting those lines, *i. e.*, a family of quadric surfaces, together with all the generators of one system on each surface, and two generators of the other system which are common to all the surfaces of the family. One of this family of quadrics we shall choose as our invariant surface. Therefore, from previous work,* we have a one-parameter group which we shall designate as ${}^{10}G_1(ll'\mu F)$, neglecting all the surfaces of the system, save the one we call F. The one-dimensional transformations are identical along l and l', and loxodromic along each of the generators of μ.

6. Of these one-parameter groups we shall find $2\infty^2$, one for each pair of generators of the same system. Hence, there are $2\infty^3$ transformations of space of type X which leave invariant a given quadric

* This reference is to unpublished work by Professor Newson.

surface, each of which leaves also invariant all the generators of one system and two generators of the other.

7. *Higher groups.*—Let us take the resultant of two transformations, ${}^{10}T(ll'\mu F)$ and ${}^{10}T_1(ll'_1\mu F)$, in which the invariant figures are the same with the exception that l'_1 is a distinct generator from l', though still of the same system. Along each of the generators l, l', l'_1 the resultant of two identical transformation is identical, while along each of the generators of μ, the resultant of two loxodromic transformations with one common invariant point is loxodromic. The remainder of the invariant figure is common and the resultant space collineation is of type X. The generator l' has been given one degree of freedom without destroying the group property. Therefore, there is a two-parameter group ${}^{10}G_2(l\mu F)$ in which l' varies over the invariant surface in the system λ.

Further, let us consider two transformations ${}^{10}T(ll'\mu F)$ and ${}^{10}T(l_1l'_1\mu F)$, where l and l' are generators of the system λ but distinct from l_1, l'_1. Each of these transformations leaves invariant every generator of μ, and must therefore transform each generator of λ into another of the same system, *i. e.*, changes a straight line cutting all the generators of μ into another straight line cutting all the generators of μ. The resultant of two loxodromic one-dimensional transformations is generally loxodromic. Moreover, since by each component the generators of λ as a system remain unchanged, the resultant also leaves the system λ unchanged, and the two new invariant lines are generators of the system λ. Hence, the resultant space collineation is of type X, and we have established the group property for a three-parameter group, ${}^{10}G_3(\mu F)$.

Furthermore, it is readily seen that there can exist no other groups of type X which leave invariant a non-degenerate quadric surface.

THEOREM. There are $2\infty^3$ transformations of space of type X which leave invariant a given quadric surface, each of which leaves also invariant all generators of one and two generators of the other system on the surface. These combine to form one-, two- and three-parameter groups but no groups of higher order.

§ 3. TYPE IX.

8. *Fundamental group.*—The fundamental invariant figure of type IX, third class, is a pencil of planes, a pencil of conics in each plane, together with a line of invariant points, and a pencil of invariant lines in one of the invariant planes. This can mean only a system of quadric surfaces to each of which the plane determined by the line of invariant points is a tangent. There are ∞^1 transformations which

leave this same figure invariant and form a one-parameter group. We shall choose as our invariant surface one of those quadrics which is tangent to p, the plane of the invariant points, and has this line as a tangent line. This group we call ${}^{9}G_1(\mathrm{lmp}t\mathrm{F})$.

The generators l and m, in which the tangent plane cuts the quadric, are rays of the invariant pencil in that plane. Therefore, along each of them there is a parabolic transformation, differing from each other only by a constant factor. Moreover, these two generators are harmonically separated by the axis of the pencil of invariant points; for, in the general case, two reciprocal polars with regard to a quadric surface are harmonically separated by that surface. But should those two polars, as a special case, intersect, they must do so in a tangent plane, and their point of intersection is on the surface. But the two axes referred to above form just this special case of reciprocal polars; and the lines harmonically separating them are the generators cut from the quadric by the tangent plane. Hence, these two axes are harmonically separated by the two invariant generators of the surface.

9. For each pair of generators we may have ∞^1 positions of the tangent line of invariant points, *i. e.*, ∞^1 of the one-parameter groups just discussed. Hence, there are of space ∞^4 collineations of type IX which leave invariant a given quadric surface, each of which leaves also invariant one generator of each system on the quadric and a line of invariant points tangent to the quadric at the intersection of these two generators.

10. *Higher groups.*—Consider next the resultant of two transformations ${}^{9}T(\mathrm{lmp}t\mathrm{F})$ and ${}^{9}T_1(\mathrm{lmp}t_1\mathrm{F})$, where t and t_1 denote different positions of the line of invariant points in the tangent plane. Since the generators l and m are always to be harmonically divided by the axes of invariant planes and points, as t shifts in the tangent plane, the axis of the pencil of planes must shift correspondingly in the same plane. We have left in each case a system of quadrics through a point tangent to a given plane. Therefore, in each component, the quadric of which l and m are generators remains unchanged. The resultant space collineation leaves invariant the given quadric and its two generators, a new pencil of planes, a new line of points, and is again of type IX. But the line t may take ∞^1 positions in the plane and through the given point. For each position there exists, as has been shown (art. 8), a one-parameter group leaving invariant a system of quadric surfaces. Each of these one-parameter groups has in common the quadric tangent to the invariant plane along l and m. Therefore, the resultant of two transformations ${}^{9}T(\mathrm{lmp}t\mathrm{F})$ and

${}^{9}T(lmpt_1F)$ is again of type IX, and we have established a two-parameter group whose symbol is ${}^{9}G_2(lmpF)$.

For any given quadric we have ∞^2 such two-parameter groups, one group for each pair of generators chosen, one generator from each system. It may be easily shown that these do not combine to form three- or four-parameter groups.

THEOREM. There are ∞^4 transformations of type IX which leave invariant a given quadric surface, each of which, in addition, leaves invariant one generator of each system and a line of invariant points through their intersection and in the plane thus determined. These combine to form one- and two-parameter groups, but not to form three- or four-parameter groups.

§ 4. TYPE VI.

11. *Fundamental group.*—The fundamental group of third class of type VI leaves invariant a pencil of planes, two points on the axis, a line of points cutting that pencil, and two planes through that line, together with a system of quadric surfaces. Of this system we choose one surface to which the two invariant axes will be reciprocal polars. We have then a one-parameter group, ${}^{6}G_1(ll'mm'nF) = {}^{6}G_1(APpsnkF)$, for each line in space taken with its reciprocal polar.

There are ∞^5 transformations of space of type VI having the required property, ∞^1 for each line in space taken with its reciprocal polar with regard to the given quadric.

13. *Higher groups.*—Take two transformations ${}^{6}T(ll'mm'nF)$ and ${}^{6}T_1(ll'mm'_1n_1F)$ as m′ shifts along the quadric, always remaining a generator of the system μ. The resultant has been shown to be ${}^{6}T_2(ll'mm'_2n_2F)$, *i.e.*, a transformation of type VI leaving the quadric invariant. Therefore, there exists a two-parameter group ${}^{6}G_2(ll'm\overline{n}F)$. In like manner the three-parameter group ${}^{6}G_3(lm\overline{n}F)$ has been determined, and it is readily seen that there can exist no three-parameter group ${}^{6}G_3(ll'\overline{n}F)$, and no groups of higher order.

Next, considering again our one-parameter group ${}^{6}G_1(APpsnF)$, allow the line of invariant points, n, to shift through P, the pole of an invariant plane, s, which cuts the surface, and to shift in a plane tangent to the surface. Let us examine the resultant of two transformations taken one from each of two one-parameter groups with different axes of invariant points. In the invariant tangent plane we have, as plane transformations, two perspective collineations with common vertex and intersecting axes of invariant points. These necessarily result in a perspective collineation having the same vertex and axis of invariant points through the intersection of the axes of the two

components. Other portions of the invariant figures of the two space collineations correspond, and hence it is readily seen that their resultant is of type VI, with a new line of invariant points still in the tangent plane and through the fixed point P. We have, then, a new two-parameter group ${}^6G_2(APpsF)$. Again, letting the line of invariant points vary through the invariant pole in space, we have a three-parameter group ${}^6G_3(PsF)$ leaving invariant pole and polar plane with regard to the invariant surface. There are ∞^3 such three-parameter groups, one for each point in space.

This would indicate ∞^6 transformations of type VI leaving invariant a quadric surface. But if we examine more closely, we see that all three-parameter groups whose poles are collinear contain one common ${}^6G_1(APps\mathit{n}F)$. Hence, there are only ∞^4 distinct one-parameter groups, or ∞^5 distinct trnsformations of type VI, leaving a quadric invariant, as before stated.

Hence, there are of type VI two varieties of two- and two varieties of three-parameter groups, containing, in all, ∞^5 collineations of space which transform a quadric surface into itself.

THEOREM. There are ∞^5 collineations of space of type VI leaving invariant a given quadric surface, each of which also leaves invariant two generators of each system on the surface, and a line of invariant points cutting the surface. These ∞^5 collineations combine to form one-parameter groups, two varieties of two- and two varieties of three-parameter groups, but no groups of higher orders.

§ 5. TYPE III.

14. *Fundamental group of first class.*—The group $G_2(ABll')_n$ of type III leaves invariant a family of quadric surfaces, and, in each, two generators of one and one generator of the other system.* In all of our work we take into consideration but one quadric surface, and shall therefore designate this group as ${}^3G_2(ll'mF)$. The one-dimensional transformations of this type are loxodromic along m, and parabolic along l and l′, while the plane collineations are of type II in each of the invariant planes.

15. Of these two-parameter groups in space we shall find ∞^3, since l, l′ and m have each one possible degree of freedom. Hence, there are ∞^5 such collineations of type III.

16. *Higher groups.*—If we consider the resultant of two transformations ${}^3T(ll'mF)$ and ${}^3T_1(ll'_1mF)$, along m the resultant of two loxodromic transformations with one common invariant point is loxodromic, along l the resultant of two parabolic transformations

* Kansas University Quarterly, vol. X, p. 95.

with common invariant point is parabolic. The resultant is of type III, and we have a three-parameter group ${}^3G_3(lmF)$ established. If, however, we allow m to vary, along l and l′, the resultant of two parabolic transformations without common invariant point is in general loxodromic. Hence, we can have no group ${}^3G_3(ll'mF)$.

Next consider ${}^3T(ll'mF)$ and ${}^3T_1(l_1l'_1mF)$. Along m the resultant of two loxodromic, one-dimensional transformations without common invariant point is loxodromic, while along the new l and l′ the two parabolic transformations give a parabolic transformation. The entire resultant is of type III, and we have established the four-parameter group ${}^3G_4(mF)$.

The resultant of any two collineations of type III, taken in more general position, is of type I. Therefore, there are no groups of higher orders.

THEOREM. There are of space ∞^4 collineations of type III which leave invariant a given quadric, and each of which leaves invariant, in addition, two generators of one and one of the other system on the surface. These collineations combine to form two-, three- and four-parameter groups of the first class.

17. *Second class.*—In the collineations of our two-parameter group ${}^3G_2(ll'mF)$, let us assume the relation between the t along l and the k along m to be $k = a^t$, where a is a constant; and take the resultant of two transformations having different t's. Along l and l′ these combine by the law $t_2 = t + t_1$, and along m according to the law $k_2 = kk_1 = a^t a^{t_1} = a^{t+t_1}$. Therefore, the a of the resultant transformation is the same as in each of its components. There is a one-parameter subgroup of ${}^3G_2(ll'mF)$ for each value of a, the symbol being ${}^3G_1(ll'mF)_a$.

18. *Higher groups.*—Take two collineations, ${}^3T(ll'mF)_a$ and ${}^3T_1(ll'_1mF)_a$, and form their resultant. In the plane lm the plane transformations are of type II, and have in common two lines and their point of intersection. But this is the case in which the plane collineations with constant a combine according to the law $t_2 = t + t_1$. Hence, the resultant of the space transformations is of type III with constant a, and there is determined a group ${}^3G_2(lmF)_a$. It may be easily seen that there are no higher groups of type III of this class.

THEOREM. There are ∞^5 collineations of type III which transform a given quadric surface into itself. These combine to form one- and two-parameter groups of the second class; but there are no groups of this class of higher orders.

§ 6. Type I.

19. *Fundamental group of first class.*—The group $G_2(ABCD)r + r' = 1$, of type I, second class, leaves invariant a tetrahedron and a system of quadric surfaces, of which AC, BD, AD and BC are generators, while CD and AB are reciprocal polars.* The one-dimensional transformations are all loxodromic, while the plane transformations are of type I. Here, again, we concern ourselves with but one of the quadric surfaces, and shall therefore designate this group as ${}^1G_2(ll'mm'F)$. This group has as its independent parameters the cross-ratios k and k'. Hence, it is a two-parameter group and contains ∞^2 collineations.

There exists one such two-parameter group for each set of four generators, chosen two from each system on the quadric. Therefore, there are ∞^6 collineations of space of type I which leave a given quadric invariant.

20. *Higher groups.*—Let us form the resultant of two transformations ${}^1T(ll'mm'F)$ and ${}^1T_1(ll'mm'_1F)$, where m' varies so as to remain always a generator of the system μ.

Express the two tetrahedrons as ABCD and AB_1CD_1, respectively. Substituting the relation of cross-ratios necessary for the presence of an invariant quadric, viz., $r + r' = 1$, we have our cross-ratios as follows:

$$
\begin{array}{rcccccc}
{}^1T = & AB & BC & CD & DB & AC & AD, \\
 & k & k^{-r} & k^{2r-1} & k^{1-r} & k^{1-r} & k^r \\
{}^1T_1 = & AB_1 & B_1C & CD_1 & D_1B_1 & AC & AD, \\
 & k_1 & k_1^{-r} & k_1^{2r-1} & k_1^{1-r} & k_1^{1-r} & k_1^r \\
{}^1T_2 = {}^1T \cdot {}^1T_1 = & AB_2 & B_2C & CD_2 & D_2B_2 & A_2C & A_2D_2, \\
 & k_2 & k_2^{-r} & k_2^{2r-1} & k_2^{1-r} & k_2^{1-r} & k_2^r
\end{array}
$$

The above resultants are obtained as follows: Along AC, two loxodromic transformations with two common invariant points combine according to the law $k_2 = kk_1$. Along AD_2 and AB_2, where the transformations have but one common invariant point, the same law holds true. Moreover, through A, in the plane ADC, the loxodromic one-dimensional collineations have two invariant rays in common, and hence, combine according to the above law, and along any line cutting that pencil of rays the new cross-ratio is k_2^{2r-1}. The two remaining cross-ratios may be determined by taking the product of cross-ratios around the various triangles, this product around any given triangle always equaling 1. From this it is readily seen that the group property holds throughout. Hence, there is a three-parameter group ${}^1G_3(ll'mF)$.

*Kansas University Quarterly, vol. X, p. 38.

This method of proof is a very general one, and is made use of in showing the existence of higher groups of this class. There exist four such groups, ${}^1G_4(ll'F)$, ${}^1G_4(lmF)$, ${}^1G_5(lF)$, and ${}^1G_6(F)$. The proof of the group property in these last cases is left to the reader.

21. *Groups of second class.*—If, to the conditions already placed on the relations of cross-ratios in the preceding groups, we add that of a constant r, we obtain several new groups.

The first of these is a one-parameter group ${}^1G_1(ll'mm'F)_r$, whose existence as a subgroup of ${}^1G_2(ll'mm'F)$ is evident.* The higher groups are as easily shown, the existence or non-existence of the group depending upon the corresponding group of the plane.† The details of the work are omitted for the two remaining groups, ${}^1G_2(lmF)_r$ and ${}^1G_3(lmF)_r$.

THEOREM. There are ∞^6 collineations of space of type I which leave invariant a given quadric surface, each of which, in addition, leaves invariant a pair of generators from each system on the quadric. These combine to form groups of two classes. Of the first class there are two-, three-, two varieties of four-, five- and six-parameter groups. Of the second class there are one-, two- and three-parameter groups, but none of the higher order.

B.—Combinations of Various Types.

§ 1. TYPE XI.

22. The resultant of two transformations of the same type we have often found to be of a different type. We shall proceed next to examine not only the resultant of two transformations of one type, but also the resultant of a collineation of one type taken with one from each of the other types. Generally, this resultant is at once evident and will be stated without proof.

The resultant of two transformations of type XI with common invariant figure was shown to be of type XI. If, however, the invariant figures differ in the generators of the second system, the resultant one-dimensional transformation is not parabolic but loxodromic, and the space collineation is of type X.

Similarly, taken with different invariant systems of generators, the resultant is of type IX.

$$ {}^{11}T(l\mu F) + {}^{11}T_1(l\mu F) = {}^{11}T_2(l\mu F) $$
$$ {}^{11}T(l\mu F) + {}^{11}T_1(l_1\mu F) = {}^{10}T(l_2 l'_2 \mu F) $$
$$ {}^{11}T(l\mu F) + {}^{11}T_1(m\lambda F) = {}^{9}T(lmtF). $$

---

* Kansas University Quarterly, vol. X, p. 33.

† American Journal of Mathematics, vol. XXIV, No. 2, p. 134.

§ 2. Type X.

23. For combinations with collineations of the same type in the case of type X, the work has been given, save that in which the invariant systems are different. This evidently gives a collineation of type I.

In combination with transformations of type XI, only one is worthy of special mention. ${}^{10}\mathrm{T}(ll'\mu\mathrm{F})$ and ${}^{11}\mathrm{T}(m\lambda\mathrm{F})$ give along l and l' the combination of identical and parabolic one-dimensional transformations, evidently a parabolic resultant, while along m the resultant of loxodromic and identical gives a loxodromic transformation. The space collineation is of type III.

The following statement is given for reference:

$$
\begin{aligned}
{}^{10}\mathrm{T}(ll'\mu\mathrm{F}) + {}^{10}\mathrm{T}_1(mm'\lambda\mathrm{F}) &= {}^{1}\mathrm{T}_2(ll'\mathrm{mm'F})\\
{}^{10}\mathrm{T}(ll'\mu\mathrm{F}) + {}^{11}\mathrm{T}(l\mu\mathrm{F}) &= {}^{10}\mathrm{T}_1(ll'_1\mu\mathrm{F})\\
{}^{10}\mathrm{T}(ll'\mu\mathrm{F}) + {}^{11}\mathrm{T}(l_1\mu\mathrm{F}) &= {}^{10}\mathrm{T}_1(l_2l'_2\mu\mathrm{F})\\
{}^{10}\mathrm{T}(ll'\mu\mathrm{F}) + {}^{11}\mathrm{T}(m\lambda\mathrm{F}) &= {}^{3}\mathrm{T}(\mathrm{ll'mF}).
\end{aligned}
$$

§ 3. Type IX.

24. It is worthy of notice that a transformation of type IX, in combination with those of other types, may result in one of type III. The resultants of type I are not very frequent in this case.

$$
\begin{aligned}
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{9}\mathrm{T}_1(\mathrm{lmp}t\mathrm{F}) &= {}^{9}\mathrm{T}_2(\mathrm{lmp}t\mathrm{F})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{9}\mathrm{T}_1(\mathrm{lmp}t_1\mathrm{F}) &= {}^{9}\mathrm{T}_2(\mathrm{lmp}t_2\mathrm{F})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{9}\mathrm{T}_1(\mathrm{lm_1p_1}t_1\mathrm{F}) &= {}^{1}\mathrm{T}(\mathrm{ll'_2m_2m'_2F})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{9}\mathrm{T}_1(\mathrm{l_1m_1p_1}t_1\mathrm{F}) &= {}^{1}\mathrm{T}(\mathrm{l_2l'_2m_2m'_2F})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{11}\mathrm{T}(\mathrm{l}\mu\mathrm{F}) &= {}^{9}\mathrm{T}_1(\mathrm{lmp}t_1\mathrm{F})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{11}\mathrm{T}(\mathrm{l_1}\mu\mathrm{F}) &= {}^{3}\mathrm{T}(\mathrm{l_2l'_2mF})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{10}\mathrm{T}(\mathrm{ll'}\mu\mathrm{F}) &= {}^{3}\mathrm{T}(\mathrm{ll'_2mF})\\
{}^{9}\mathrm{T}(\mathrm{lmp}t\mathrm{F}) + {}^{10}\mathrm{T}(\mathrm{l_1l'_1}\mu\mathrm{F}) &= {}^{3}\mathrm{T}(\mathrm{l_2l'_2mF}).
\end{aligned}
$$

§ 4. Type VI.

25. The resultant of a transformation of type VI with another of the same type has been discussed. Combined with another of any other type, the resultant is easily seen to be always of type I, since no transformation of another type leaves invariant all points on a polar line, save a special transformation of type X, which will be treated under the head of singular transformations,* or when combined with one of type IX. This last case we will explain more fully. Take two transformations, ${}^{6}\mathrm{T}(\mathrm{ll'mm'}n\mathrm{F})$ and ${}^{9}\mathrm{T}(\mathrm{l'mp}t\mathrm{F})$. On the quadric

* Art. 35.

surface the resultant transformation evidently leaves invariant the figure $(l_2l'mm'_2)$. In the plane of $l'm$ we have the resultant of an elation and perspective collineation, the axis of the elation passing through the vertex of the prospective collineation. This is a perspective collineation having the same vertex and axis as the first. The complete invariant figure of the space collineation is $(l_1l'm_1m'nF)$, which is that of type VI. Taken in any other position, the two component transformations result in one of type I.

§ 5. Type III.

26. The combinations of type III, first class, are comparatively simple. But one thing is to be noted: the resultant of a transformation with one of any other type is, in general, never anything other than type II or type I; but type III may be combined with any other type to produce a transformation of type I.

$$
\begin{aligned}
{}^3T(ll'mF) + {}^3T_1(ll'mF) &= {}^3T_2(ll'mF)\\
{}^3T(ll'mF) + {}^3T_1(ll'_1mF) &= {}^3T_2(ll'_2mF)\\
{}^3T(ll'mF) + {}^3T_1(l_1l'_1mF) &= {}^3T_2(l_2l'_2mF)\\
{}^3T(ll'mF) + {}^3T_1(l_1l'_1m_1F) &= {}^1T(l_2l'_2m_2m'_2F)\\
{}^3T(ll'mF) + {}^3T_1(ll'm_1F) &= {}^1T(ll'm_2m'_2F)\\
{}^3T(ll'mF) + {}^3T_1(ll'_1m_1F) &= {}^1T(ll'_2m_2m'_2F)\\
{}^3T(ll'mF) + {}^{11}T(l\,\mu\,F) &= {}^3T_1(ll'_1mF)\\
{}^3T(ll'mF) + {}^{11}T(l_1\,\mu\,F) &= {}^3T_1(l_2l'_2mF)\\
{}^3T(ll'mF) + {}^{11}T(lm\,\lambda\,F) &= {}^3T_1(ll'mF)\\
{}^3T(ll'mF) + {}^{11}T(m'\,\lambda\,F) &= {}^1T(ll'm_2m'_2F)\\
{}^3T(ll'mF) + {}^{10}T(ll'\,\mu\,F) &= {}^3T_1(ll'mF)\\
{}^3T(ll'mF) + {}^{10}T(ll'_1\,\mu\,F) &= {}^3T_1(ll'_2mF)\\
{}^3T(ll'mF) + {}^{10}T(l_1l'_1\,\mu\,F) &= {}^3T_1(l_2l'_2mF)\\
{}^3T(ll'mF) + {}^{10}T(mm'\,\lambda\,F) &= {}^1T(ll'mm'_2F)\\
{}^3T(ll'mF) + {}^{10}T(m_1m'\,\lambda\,F) &= {}^1T(ll'm_2m'_2F)\\
{}^3T(ll'mF) + {}^9T(lmpt F) &= {}^3T_1(ll'_2mF)\\
{}^3T(ll'mF) + {}^9T(lmpt F) &= {}^3T_1(ll'_2m_2F)\\
{}^3T(ll'mF) + {}^9T(l_1mp_1t_1F) &= {}^1T(l_2l'_2mm'_2F).
\end{aligned}
$$

§ 6. Type I.

27. The resultant of a transformation of type I with any other transformation is, in general, of type I. Moreover, a transformation of type I may be resolved into components of any one of the five types discussed in this paper.

In all the work of the preceding articles, it must be borne in mind

that the values of the cross-ratios of the one-dimensional transformations in each case are taken perfectly general and with no fixed relation to each other. Our next task will be to investigate the result of choosing space collineations in which the one-dimensional transformations bear definite relations to each other. By proper choice of parameters, we may resolve a collineation of any type into components of almost any other type. This is the foundation principle of group structure; for it is only by this means that we may obtain groups of one type within groups of another type.

C.—Group Structure.

§ 1. Type XI.

28. For special values or specially related values of the parameters of a group, the resulting transformation may degenerate into one of a lower type. For the most part these will be only indicated where they are easily seen. It has been shown already that the group ${}^{11}G_1(l\,\mu\,F)$ contains only transformations of that type. Hence we may state in symbolic language as its structure,

$$G_6(F) \doteq 2\,\infty^1\ {}^{11}G_1(l\,\mu\,F).$$

§ 2. Type X.

29. Two transformations of type X, having reciprocal values of k, the same invariant system, but only one other common invariant generator, result in a collineation of type XI; since along a line two loxodromic transformations, with reciprocal values of k and but one invariant point in common, result in a parabolic transformation. Hence, by proper choice of components within the group ${}^{10}G_2(l\,\mu\,F)$ we may get all the collineations of the group ${}^{11}G_1(l\,\mu\,F)$. Combining ∞^1 of the groups ${}^{10}G_2(l\,\mu\,F)$, we get ${}^{10}G_3(\mu\,F)$; but the ∞^1 ${}^{11}G_1(l\,\mu\,F)$ do not form a two-parameter group of type XI. Symbolically expressed, we have:

$$\begin{aligned}
{}^{10}G_1(ll'\,\mu\,F) &= {}^{10}G_1(ll'\,\mu\,F)\\
{}^{10}G_2(ll'\,\mu\,F) &= \infty^1\ {}^{10}G_1(l\,\bar{l}'\,\mu\,F) + {}^{11}G_1(l\,\mu\,F)\\
{}^{10}G_3(\mu\,F) &= \infty^2\ {}^{10}G_1(\bar{l}\,\bar{l}'\,\mu\,F) + \infty^1\ {}^{11}G_1(\bar{l}\,\mu F)\\
G_6(F) \doteq 2\,{}^{10}G_3(\bar{\mu}\,F) &= 2\,\infty^2\ {}^{10}G_1(\bar{l}\,\bar{l}'\,\bar{\mu}\,F) + 2\,\infty^1\ {}^{11}G_1(\bar{l}\,\bar{\mu}\,F).
\end{aligned}$$

§ 3. Type IX.

30. In article (10) it was shown that the invariant generators of a transformation of type IX divided harmonically the line of invariant points and the axis of invariant planes. Therefore, as t varies, the axis of planes must also vary, so that when t, the axis of invariant

points, coincides with one of the invariant generators, it must also coincide with the axis of planes. In this case, however, the transformation of type IX degenerates to one of type XI. The line of invariant points may coincide with either generator giving two groups, ${}^{11}G_1(l\,\mu\,F)$ and ${}^{11}G_1(m\,\lambda\,F)$, one on each system of generators of the quadric surface.

The resultant of two transformations of type XI, taken one from each of these groups, is of type IX (art. 22). From the $2\,\infty^1$ transformations included within these two groups may be found ∞^2 combinations of the kind discussed, *i. e.*, ∞^2 transformations of type IX which combine to form a two-parameter group, and every transformation of type IX which leaves invariant a quadric surface may be resolved into components of type XI.

$$ {}^{9}G_1(\mathrm{lmp}\,t\,F) = {}^{9}G_1(\mathrm{lmp}\,t\,F) $$

$$ {}^{9}G_2(\mathrm{lmpF}) = \infty^1\ {}^{9}G_1(\mathrm{lmp}\,\bar{t}\,F) + {}^{11}G_1(l\,\mu\,F) + {}^{11}G_1(m\,\lambda\,F) $$

$$ G_6(F) \equiv \infty^2\ {}^{9}G_2(\mathrm{lmpF}) = \infty^3\ {}^{9}G_1(\mathrm{lmp}\,t\,F) + 2\,\infty^1\ {}^{11}G_1(l\,\mu\,F). $$

§ 4. Type VI.

31. The presence of a group ${}^{11}G_1(l\,\mu\,F)$ in the two-parameter group ${}^{6}G_2(\mathrm{ll'm}\,\bar{n}\,F)$ is determined as in the case of type X (art. 29). The only other groups of different type are those of type IX. In the group ${}^{6}G_3(\mathrm{PsF})$ the line of invariant points shifts through the pole P in space. It may take ∞^1 positions tangent to the quadric surface; but in these cases the transformation degenerates to type IX. Hence there are ∞^1 one-parameter groups of type IX included in the group ${}^{6}G_3(\mathrm{PsF})$. These do not form a two-parameter group ${}^{9}G_2(\mathrm{lmpF})$, since the line of invariant points shifts, not in a plane and through a point on the surface, but about the surface of a cone with vertex not on the quadric. Therefore, the resultant of two transformations taken one from each of two one-parameter groups is not of type IX.

$$ {}^{6}G_1(\mathrm{ll'mm'}\,n\,F) = {}^{6}G_1(\mathrm{APps}\,n\,F) + \mathrm{S.T.} $$

$$ {}^{6}G_2(\mathrm{ll'm}\,n\,F) = \infty^1\ {}^{6}G_1(\mathrm{ll'm}\,\bar{\mathrm{m}}'\,\bar{n}\,F) + {}^{11}G_1(m\,\lambda\,F) + \mathrm{S.T.} $$

$$ {}^{6}G_3(\mathrm{lm}\,\bar{n}\,F) = \infty^2\ {}^{6}G_1(\mathrm{ll'm}\,\bar{\mathrm{m}}'\,\bar{n}\,F) + {}^{9}G_2(\mathrm{lmpF}) + {}^{11}G_1(l\,\mu\,F) + {}^{11}G_1(m\,\lambda\,F) + \mathrm{S.T.} $$

$$ {}^{6}G_2(\mathrm{APpsF}) = \infty^1\ {}^{6}G_1(\mathrm{APps}\,\bar{n}\,F) + {}^{9}G_1(\mathrm{lmp}\,t\,F) + \mathrm{S.T.} $$

$$ {}^{6}G_3(\mathrm{PsF}) = \infty^2\ {}^{6}G_1(\mathrm{AP}\,\bar{\mathrm{p}}\,\mathrm{s}\,\bar{n}\,F) + \infty^1\ {}^{9}G_1(\mathrm{l}\,\bar{\mathrm{m}}\,\bar{\mathrm{p}}\,\bar{t}\,F). $$

§ 5. Type III.

32. If from the group ${}^{3}G_2(ll'mF)$ we choose two transformations with reciprocal values of k, then along m the result is identical, and we have the invariant figure $(m\lambda F)$, that of type XI. For every pair of reciprocal values of k, we have a new transformation of type XI with this same invariant figure; hence the group ${}^{11}G_1(m\lambda F)$. Similarly we determine the presence of the group ${}^{10}G_1(ll'\mu F)$.

In combining ∞^1 of these two-parameter groups of type III, we find that we have not ∞^1 distinct groups ${}^{11}G_1(m\lambda F)$, but only the same one occurring each time. There now appears another group ${}^{11}G_1(l\mu F)$, each tranformation of which is obtained by choosing two transformations from different two-parameter groups ${}^{3}G_2(ll'mF)$ and ${}^{3}G_2(ll''mF)$, with $t = -t_1$, and $k = \frac{1}{k_1}$, when our transformation along l becomes identical, and that along m parabolic. Having these two groups, ${}^{11}G_1(l\mu F)$ and ${}^{11}G_1(m\lambda F)$, on different systems of generators, the presence of ${}^{9}G_2(lmpF)$ is recognized from the previous work (art. 30), and we have, finally:

$$ {}^{3}G_2(ll'mF) = {}^{3}G_2(ll'mF) + {}^{10}G_1(ll'\mu F) + {}^{11}G_1(m\lambda F). $$

$$ {}^{3}G_3(lmF) = \infty^1\ {}^{3}G_2(ll'mF) + {}^{10}G_2(l\mu F) + {}^{9}G_2(lmpF) + {}^{11}G_1(l\mu F) + {}^{11}G_1(m\lambda F). $$

$$ {}^{3}G_4(mF) = \infty^2\ {}^{3}G_2(\bar{l}\bar{l}'mF) + {}^{10}G_3(\mu F) + \infty^1\ {}^{9}G_2(\bar{l}m\bar{p}F) + \infty^1\ {}^{11}G_1(\bar{l}\mu F) + {}^{11}G_1(m\lambda F). $$

It is to be noted that in ${}^{3}G_4(mF)$, while there occurs $\infty^1\ {}^{11}G_1(l\mu F)$, the group ${}^{11}G_1(m\lambda F)$ occurs only once.

Second class.—The structure of the second class involves but one type other than itself, save where transformations of such types occur as singular transformations. If we consider the resultant of two transformations of type III, with l and m as the only common invariant generators, we have the resultant one-dimensional transformations parabolic; *i. e.*, the resultant space collineation is of type IX.

$$ {}^{3}G_1(ll'mF)_a = {}^{3}G_1(ll'mF)_a + \text{S. T.} $$

$$ {}^{3}G_2(lmF)_a = \infty^1\ {}^{3}G_1(l\bar{l}'mF)_a + {}^{9}G_1(lmptF) + \text{S. T.} $$

§ 6. Type I.

33. *First class.*—The groups of this most general type include all of the groups of the types heretofore considered. The method of determining the presence of the various types is the same as that employed before. The list, while long, is very simple, if we bear in mind that, having once determined the presence of one group, we have necessarily determined at the same time the presence of all of its component parts. Thus, for example, in ${}^{1}G_3(ll'mF)$, we may de-

termine the existence of ${}^{3}G_2(ll'm)$, by taking two tranformations having on l and l′ one common invariant point with reciprocal values of k, the resultant is at once a parabolic transformation, and the space collineation is of type III. If we combine ∞^1 of these, in order to produce ${}^{1}G_4(lmF)$, we have also combined ∞^1 ${}^{3}G_2(ll'mF)$, giving us the group ${}^{3}G_3(lmF)$. But this group we have already proved to contain ${}^{9}G_2(lmpF)$, ${}^{11}G_1(l\mu F)$, and ${}^{11}G_1(m\lambda F)$. Hence, we may state without further comment that these same three groups exist in ${}^{1}G_4(lmF)$.

The existence in the fundamental group ${}^{1}G_2(ll'mm'F)$ of both of the dualistic groups ${}^{6}G_1(ll'mm'nF)$ and ${}^{6}G_1(ll'mm'n'F)$ of type VI, and of the two one-parameter groups of type X, one on each system of generators of the invariant surface, is apparent, the conditions necessary to produce them being merely the proper choice of the parameters r and r′ of the group. The following table gives the group structure of type I, first class, in symbolic language:

$$
\begin{aligned}
{}^{1}G_2(ll'mm'F) \equiv {}&{}^{1}G_2(ll'mm'F) + {}^{10}G_1(ll'\mu F) + {}^{10}G_1(mm'\mu F) \\
&+ {}^{6}G_1(ll'mm'nF) + {}^{6}G_1(ll'mm'n'F).
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}G_3(ll'mF) \equiv {}&\infty^1\ {}^{1}G_2(ll'm\overline{m}'F) + {}^{3}G_2(ll'mF) + {}^{6}G_2(ll'm\overline{n}F) \\
&+ {}^{6}G_2(ll'm\overline{n}'F) + {}^{10}G_1(ll'\mu F) + {}^{10}G_2(m\lambda F) \\
&+ {}^{11}G_1(m\lambda F).
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}G_4(lmF) = {}&\infty^2\ {}^{1}G_2(ll'mm'F) + {}^{3}G_3(lmF) + {}^{6}G_3(lm\overline{n}F) \\
&+ {}^{6}G_3(ApF) + {}^{10}G_2(l\mu F) + {}^{10}G_2(m\lambda F) \\
&+ {}^{11}G_1(m\lambda F) + {}^{11}G_1(l\mu F) + {}^{9}G_2(lmpF).
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}G_4(ll'F) = {}&\infty^2\ {}^{1}G_2(ll'\overline{m}\,\overline{m}'F) + \infty^1\ {}^{3}G_2(ll'\overline{m}F) + \infty^1\ {}^{6}G_2(ll'\overline{m}\,\overline{n}F) \\
&+ \infty^1\ {}^{6}G_2(ll'\overline{m}\,\overline{n}'F) + {}^{10}G_1(ll'\mu F) + {}^{10}G_3(\lambda F) \\
&+ \infty^1\ {}^{11}G_1(\overline{m}\lambda F).
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}G_5(lF) = {}&\infty^3\ {}^{1}G_2(l\overline{l}'\overline{m}\,\overline{m}'F) + {}^{3}G_4(lF) + \infty^1\ {}^{6}G_3(l\overline{m}\,\overline{n}F) \\
&+ \infty^1\ {}^{6}G_3(A\overline{p}F) + {}^{10}G_2(l\mu F) + {}^{10}G_3(\lambda F) \\
&+ {}^{11}G_1(l\mu F) + \infty^1\ {}^{11}G_1(m\lambda F) + \infty^1\ {}^{9}G_2(l\overline{m}\,\overline{p}F).
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}G_6(F) \quad {}&\infty^4\ {}^{1}G_2(\overline{l}\,\overline{l}'\overline{m}\,\overline{m}'F) + 2\,\infty^1\ {}^{3}G_4(\overline{l}F) + \infty^2\ {}^{6}G_3(\overline{l}\,\overline{m}\,\overline{n}F) \\
&+ \infty^2\ {}^{6}G_3(\overline{A}\,\overline{p}F) + \infty^3\ {}^{6}G_3(\overline{P}\,\overline{s}F) + 2\ {}^{10}G_3(\overline{\lambda}F) \\
&+ 2\,\infty^1\ {}^{11}G_1(\overline{m}\overline{\lambda}F) + \infty^2\ {}^{9}G_2(\overline{l}\,\overline{m}\,\overline{p}F).
\end{aligned}
$$

34. *Second class.*—We found (art. 21) that for any definite value of r there existed a one-parameter subgroup of ${}^{1}G_2(ll'mm'F)$. But with r a rational number, we may have three cases, viz., r may be odd over odd, even over odd, odd over even; the fourth possible case, even over even, degenerating always to some one of the other three. The group structure in each of these three cases must be separately examined. For all continuous groups the three cases are identical; but we shall see that, with regard to singular transformations, the three are distinct. Hence, in our present statement, we shall merely indicate

the structure in a general way, leaving the fuller discussion of the three varieties as distinct groups for the next article.

The general group structure will be seen to be identical with the first three groups of type VI, which is in fact only the special case of type I, for $r = \infty$.

$$^1G_1(ll'mm'F)_r = {}^1G_1(ll'mm'F)_r + \text{S. T.}$$

$$^1G_2(ll'mF)_r = \infty^1\ {}^1G_1(ll'm\overline{m}'F)_r + {}^{11}G_1(m\lambda F) + \text{S. T.}$$

$$^1G_3(lmF)_r = \infty^2\ {}^1G_1(l\overline{l}m\overline{m}'F)_r + {}^{11}G_1(m\lambda F) + {}^{11}G_1(l\mu F) + {}^9G_2(lmpF) + \text{S. T.}$$

D.—Singular Transformations.

§ 1. Type I.

35. We shall define as a singular transformation one of a discrete set of transformations of a lower type occurring in a continuous group of a higher type. Such transformations are not found in the groups of the types IX, X, or XI, nor in the groups of the first class of types I or III.

If in $^1G_2(ll'mm'F)$ we choose our two parameters k and r, such that r is any rational number and $k = \pm 1$, we have a transformation of type X; for if we choose our tetrahedron to be ABCD, then the cross-ratios are:

AB	BC	CD	BD	AC	AD
k	k^{-r}	$k^{r-r'}$	$k^{-r'}$	k^{1-r}	$k^{1-r'}$,

and since $r - r' = 1$, the condition for invariant quadric surface, we have, after making $k = -1$ and $r = \frac{o}{o}$, then $r = \frac{e}{o}$,

AB	BC	CD	BD	AC	AD
-1	-1	$+1$	$+1$	$+1$	-1 for $r = \frac{o}{o}$
-1	$+1$	-1	-1	-1	$+1$ for $r = \frac{e}{o}$,

If we choose $k = -1$ when $r = \frac{o}{e}$ we have imaginary cross-ratios. But if $r = \frac{e}{o}$ and $k = +1$, we have the following set of ratios:

AB	BC	CD	BD	AC	AD
$+1$	-1	$+1$	-1	-1	-1.

The only interpretation of these ratios, however, is a transformation of type X with the axes of invariant points, reciprocal polars with regard to the invariant quadric. A transformation of type X with general cross-ratios cannot leave invariant a quadric surface, with the condition just stated; but, for every pair of reciprocal polars with regard to the given quadric, there exists one involutoric collineation which does leave that quadric invariant. Hence, there are ∞^4 involutoric transformations of space of type X having the axes of invariant points

reciprocal polars with regard to the quadric surface which they leave invariant.

In each ${}^1G_1(ll'mm'F)_r$, where r is a rational number, there exists one of these collineations of type X, which we shall therefore term a singular transformation. The value of k and the position of the axes in each case depends upon the value of r—*i. e.*, whether it be odd over odd, even over odd, odd over even. If we combine ∞^1 of the one-parameter groups of type I, where r is odd over odd, we obtain no group of type X; nor, if we combine each of the singular transformations with the transformations of type I of the group, or with those of type XI occurring in this group, do we obtain any new singular transformations. However, if r be even over odd, we see (art. 35) that the group of type XI and the singular transformation of type X have their invariant systems lying in opposite directions. But a combination of a collineation of type X and one of type XI in this position we have shown (art. 23) to be of type III. Each singular transformation of type X combined with each of the group of collineations of type XI gives such a resultant. These resultants do not form a continuous group, but there exists a discrete set of them, and they must be classed as singular transformations of type III. In the third case, r being odd over even, the axes of invariant points are reciprocal polars with regard to the given quadric, and the combination of this transformation with those of types I, IX, XI, gives only type I.

For singular transformations in type I, we may then state in symbolic language:

$${}^1G_1(ll'mm'F)_r = \frac{o}{o} \overline{\cdot} \; {}^{10}S.T.(mm'\lambda F)_{k=-1}$$

$${}^1G_2(ll'mmF)_r = \frac{o}{o} \overline{\cdot} \; \infty^1 \; {}^{10}S.T.(m\,\overline{m}'\lambda F)_{k=-1}$$

$${}^1G_3(lmF)_r = \frac{o}{o} \overline{\cdot} \; \infty^1 \; {}^{10}S.T.(m\,\overline{m}'\lambda F)_{k=-1}$$

$${}^1G_1(ll'mm'F)_r = \frac{e}{o} \overline{\cdot} \; {}^{10}S.T.(ll'\mu F)_{k=-1}$$

$${}^1G_2(ll'mF)_r = \frac{e}{o} \overline{\cdot} \; \infty^1 \; {}^{10}S.T.(ll'\mu F)_{k=-1} + {}^3S.T.(ll'mF).$$

$${}^1G_3(lmF)_r = \frac{e}{o} \overline{\cdot} \; \infty^1 \; {}^{10}S.T.(l\,\overline{l}\,\mu F)_{k=-1} + \infty^1 \; {}^3S.T.(l\,\overline{l}'\,mF)$$

$${}^1G_1(ll'mm'F)_r = \frac{o}{e} \overline{\cdot} \; {}^{10}S.T.(ll'mm'nn'F)_{k=+1}$$

$${}^1G_2(ll'mF)_r = \frac{o}{e} \overline{\cdot} \; \infty^1 \; {}^{10}S.T.(ll'm\,\overline{m}'\,\overline{n}\,\overline{n}'F)_{k=+1}$$

$${}^1G_3(ll'F)_r = \frac{o}{e} \overline{\cdot} \; \infty^2 \; {}^{10}S.T.(l\overline{l}'m\,\overline{m}'\,\overline{n}\,\overline{n}'F)_{k=+1}.$$

It should be carefully noted that these transformations of type X cannot combine to form a group, since the resultant of two of them,

with the same invariant figure, would give an identical transformation, while if their invariant figures differed in any part the resultant would necessarily be of type I.

§ 2. Type VI.

36. The groups of type VI of the first variety correspond very closely to those of type I, second class. If we choose $k = -1$, the cross-ratios become:

AB	BC	CD	BD	AC	AD
$+1$	-1	$+1$	-1	-1	-1,

which we see to be an involutoric transformation of type X, with the invariant axes reciprocal polars with regard to the quadric surface. Such transformations, we have seen, cannot form a group. Moreover, in each ${}^6G_1(ll'mm'nn'F)$ there exists but one such transformation—*i. e.*, the one for $k = -1$.

In building up the higher groups of type VI of the first variety these singular transformations remain throughout, but when combined with other types which may occur in the group of type VI, they produce no new singular transformations. But in ${}^6G_2(APpsF)$ we have a group of collineations of type IX, each of which, when combined with the singular transformations of type X, gives a resultant of type IX, but with a new line of invariant points, therefore, not belonging to the group ${}^9G_1(lmptF)$—*i. e.*, we have singular transformations. Moreover, these will remain as singular transformations in ${}^6G_3(PsF)$, since the ∞^1 axes of invariant points included in the ∞^1 ${}^9G(lmptF)$ necessarily lie in a cone, while those of the singular transformation may take any position in space.

§ 3. Type III.

37. The only singular transformations left for consideration are those of type XI and type IX occurring in groups of type III, second class. Our fundamental group here is ${}^3G_1(ll'mF)_a$, in which there exists the relation of parameters, $k = a^t$ where a is a constant. It has been proved* that we may so choose two transformations with cross-ratios $k = a^t$ and $k_1 = a^t$, that the resultant may be $k_2 = a^{t+t_1} = 1$, where $t + t_1$ is not 0. Such a resultant would give an identity along m, and a parabolic transformation along each generator of system λ. This is a collineation of type XI. These singular transformations of type XI in the group ${}^3G_2(lmF)_a$ may combine with the collineation of ${}^9G_1(lmptF)$, resulting in singular transformations of type IX.

Therefore, we find:

$$ {}^3G_1(ll'mF)_a \doteq {}^{11}S.T.(m\lambda F) $$
$$ {}^3G_2(lmF)_a \doteq {}^{11}S.T.(m\lambda F) + {}^9S.T.(lmp\bar{t}F). $$

*American Journal of Mathematics, vol. XXIV, No. 2, p. 159.

E.—Mixed Groups.

§ 1. Type XII.

38. We now come to the consideration of what Doctor Wood terms* "symmetry transformations," *i. e.*, transformations which leave a given quadric surface invariant, but interchange the two systems of generators. The discussion of such transformations brings to our notice collineations of a type not yet considered. Choose as the invariant figure of a transformation of type XII the vertex and axial plane as pole and polar with regard to the given quadric surface, and next choose one cross-ratio along each invariant line equal -1. Then each invariant line cuts the surface in two points which are harmonically divided by the two invariant points on that line, since a quadric surface divides every line through a pole and polar plane harmonically. But this is merely stating, in a different way, that the cross-ratio of the four points is -1; *i. e.*, in the given transformation, the two points on the quadric surface are interchanged. An effect of the same kind is produced along each of the invariant lines; *i. e.*, every point on the surface of the quadric is interchanged with another, but the surface as a whole is transformed into itself. There are ∞^3 such transformations which leave the given quadric invariant, one for each point in space taken with its polar with regard to a given surface. The resultant of these transformations with those of the types heretofore discussed give rise to the mixed groups—*i. e.*, those groups which leave a given figure as a whole invariant, but interchange its parts in such a manner that a repetition of the transformation will bring it back to the original invariant figure.

A combination of this involutoric transformation of type XII with any one of types XI or X cannot result in a transformation of either of these types. Therefore, we can have no mixed groups of these types.

§ 2. Type IX.

39. If we consider the group ${}^9G_2$(lmpF), we see that the invariant figure consists of one generator of each system and the plane of those generators. We may choose an involutoric collineation of type XII which will interchange these two generators. Their point of intersection must be an invariant point of the involutoric transformation. For any plane through that point as axial plane and its pole as vertex of an involutoric collineation, the point must remain invariant. Thus there are ∞^2 involutoric collineations of type XII which leave the figure of ${}^9G_2$(lmpF) invariant as a whole, though interchanging parts of it. We see at once that our mixed group must

* Annals of Mathematics, second series, vol. II, No. 4, p. 167.

contain *all* of the continuous subgroups included under the head of the continuous groups of like symbol. Hence, in writing out these groups, we shall not express in full detail the structure of the continuous groups, leaving that to be understood from previous work.

$$m^9G_2(lmpF) = {}^9G_2(lmpF) + \infty^2\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}.$$

§ 3. TYPE VI.

40. The continuous group ${}^6G_1(ll'mm'nF)$ may be combined with ∞^1 involutoric collineations of type XII and the invariant figure as a whole be left unchanged; *i. e.*, there will exist one involutoric ${}^6T(PsF)$, which will interchange l and m, l′ and m′ for each plane passing through the two invariant points, the intersection of l and m, l′ and m′. These combine to form a one-parameter mixed group $m^6G_1(ll'mm'n'F)$. There can exist no $m^6G_2(ll'mF)$, since, if we combine with this a transformation which interchanges the systems of generators, l′ will be thrown into a position m′, thus giving a new invariant figure and destroying one of the essentials of the group property.

The group ${}^6G_3(lmF)$ may be combined with ∞^2 of these involutoric collineations, exactly as in the case of the group ${}^9G_2(lmpF)$, and there exists the three-parameter mixed group $m^6G_3(lmF)$.

There can exist no other mixed group of type VI, since the combination of ${}^6T(PsF)$ with ${}^{12}T(PsF)$ does not alter any part of the invariant figure.

THEOREM. There are two mixed groups of transformations of type VI, each composed of a continuous group of the same symbol and involutoric transformations of type XII. Expressed symbolically, they are:

$$m^6G_1(ll'mm'nF) = {}^6G_1(ll'mm'nF) + \infty^1\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}$$

$$m^6G_3(lm\,\overline{n}\,F) = {}^6G_3(lm\,\overline{n}\,F) + \infty^2\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}.$$

§ 4. TYPE III.

41. *First class.*—It is readily seen that there can exist but one mixed group of this type and class, there being but one corresponding continuous group having similar invariant parts on the two systems of generators. This is the continuous group ${}^3G_3(lmF)$, which, if combined with $\infty^2\ {}^{12}T(PsF)_{k=-1}$, gives us the mixed group

$$m^3G_3(lmF) = {}^3G_3(lmF) + \infty^2\ {}^{12}T(P\,\overline{s}\,F)_{k=-1}.$$

42. *Second class.*—There exists in this class a continuous group ${}^3G_2(lmF)_a$, which if combined with $\infty^2\ {}^{12}T(PsF)_{k=-1}$, results in the mixed group $m^3G_2(lmF)_a$.

$$m^3G_2(lmF)_a = {}^3G_2(lmF)_a + \infty^2\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}.$$

§ 5. Type I.

43. *First class.*—There are three continuous groups of type I, first class, having corresponding parts on each set of generators. Each of these may be combined with ${}^{12}T(PsF)_{k=-1}$, giving us a corresponding mixed group. In the first of these groups there are necessarily two invariant points, and therefore but $\infty^1\ {}^{12}T(PsF)_{k=-1}$. This gives us the mixed group $m^1G_2(ll'mm'F)$. In the second case, there is but one invariant point; hence, $\infty^2\ {}^{12}T(PsF)_{k=-1}$, and the third case drops to a continuous group.

In symbolic language:

$$m^1G_2(ll'mm'F) = {}^1G_2(ll'mm'F) + \infty^1\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}$$

$$m^1G_4(lmF) = {}^1G_4(lmF) + \infty^2\ {}^{12}T(\overline{P}.\,\overline{s}.\,F.)_{k=-1}.$$

44. *Second class.*—The second-class groups are derived from those of the first class by making the parameter r any constant value, and we have:

$$m^1G_1(ll'mm'F)_a = {}^1G_1(ll'mm'F)_a + \infty^1\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}$$

$$m^1G_3(lmF)_a = {}^1G_3(lmF)_a + \infty^2\ {}^{12}T(\overline{P}\,\overline{s}\,F)_{k=-1}.$$

F.—Table of Groups of Collineations of Space Leaving Invariant a Quadric Surface.

Symbol.	*Additional invariant figure.*	*Lie.**
	Type XI.	
(1) ${}^{11}G_1(l\mu F)$.	All generators of one system, and one generator of other system.	(VI. A)
	Type X.	
(2) ${}^{10}G_1(ll'\mu F)$.	All generators of one system, and two generators of second system.	(VI. D)
(3) ${}^{10}G_2(l\mu F)$.	All generators of one system, and one generator of other system.	(V. A)
(4) ${}^{10}G_3(\mu F)$.	All generators of one system.	(IV. A)
	Type IX.	
(5) ${}^{9}G_1(lmptF)$.	One generator of each system, their common tangent plane, a line of invariant points in that plane tangent to the quadric.	(VI. B)
(6) ${}^{9}G_2(lmpF)$.	One generator of each system and their common tangent plane.	(V. E)

* Theorie der Transformationsgruppen, vol. III, pp. 251-254.